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Short Communication

Generalized Trace Inequalities for Q Uncertainty Relations

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Abstract

Iin 2015 we obtained non-hermitian extensions of Heisenberg type and Schrödinger type uncertainty relations for generalized metric adjusted skew information or generalized metric adjusted correlation measure and gave the results of Dou-Du in 2013 and 2014 as corollaries. In this paper, we define generalized quasi-metric adjusted Q skew information for different two generalized states and obtain corresponding uncertainty relation. The result is applied to the inequalities related to fidelity and trace distance for different two generalized states which were given by Audenaert, et al. in 2009 and 2008; and Powers-Strmer in 1970.

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Introduction

In quantum mechanics, it is well known that the Heisenberg/Schrödinger uncertainty relations hold for two non-commutative observables and density operators. Dou and Du obtained several uncertainty relations for two non-commutative non-hermitian observables and density operators in [1,2]. In [3-5] we gave non-hermitian extensions of Heisenberg type or Schrödinger type uncertainty relations for the generalized metric adjusted skew information or generalized metric adjusted skew correlation measure which were obtained in Yanagi, Furuichi, and Kuriyama in [6]. In this paper, we extend the non-hermitian uncertainty relations to q-uncertainty relation and apply them to the trace inequalities related to fidelity and trace distance for different two generalized states given by Audenaert et al and Powers-Stφrmer in [7-10].

Let $M_n(\mathbb{C})$ (resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle X, Y\rangle = Tr[X^* Y]$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$. A function $f:(0,+\infty)\to\mathbb{R}$ is said operator monotone if, for any $n\in\mathbb{N}$, and $A, B \in M_{n,+}(\mathbb{C})$ such that $0 \le A \le B$, the inequality $0 \le f(A) \le f(B)$ holds. An operator monotone function is said symmetric if $f(x)=xf(x^{-1})$ and normalized if $f(1)=1$.

Definition 1.1 Let $\mathfrak{F}_{op}$ be the class of functions $f:(0,+\infty)\to(0,+\infty)$ satisfying

$$f(1)=1,$$

$$tf(t^{-1}) = f(t),$$

f is operator monotone.

For $f \in \mathfrak{F}_{op}$ define $f(0)=\lim_{x\to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathfrak{F}_{op}^r = \{f \in \mathfrak{F}_{op} \mid f(0) \neq 0\},\ \mathfrak{F}_{op}^n = \{f \in \mathfrak{F}_{op} \mid f(0) = 0\}$$

and notice that trivially $\mathfrak{F}_{op} = \mathfrak{F}_{op}^r \cup \mathfrak{F}_{op}^n$. In Kubo Ando's theory of matrix means one associates a mean to each operator monotone function $f \in \mathfrak{F}_{op}$ by the formula

$$m_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $A, B \in M_{n,+}(\mathbb{C})$. By using the notion of matrix means we define the generalized monotone metrics $X, Y \in M_n(\mathbb{C})$ by the following formula

$$\langle X, Y\rangle_{f,q} = Tr[X^* m_f(L_A, qR_B)^{-1} Y],$$

where $L_A(X) = AX, R_B(X) = XB$ and $q > 0$.

Generalized Quasi-metric adjusted Q Skew information and Q correlation measure

Definition 2.1 Let $g, f \in \mathfrak{F}_{op}^r$ satisfy

$$g(x) \ge k\frac{(x-1)^2}{f(x)}$$

for some $k > 0$. We define

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathfrak{F}_{op}. \quad (2.1)$$

Definition 2.2 Notation as in Definition 2.1. For

$X,Y \in M_n(\mathbb{C})$, $A,B \in M_{n,+}(\mathbb{C})$ and $q>0$, we define the following quantities: $\Gamma_{A,B,q}^{(g,f)}(X,Y) = k\langle (L_A - qR_B)X, (L_A - qR_B)Y\rangle_{f,q}$

$$= kTr[X^*(L_A - qR_B)m_f(L_A, qR_B)^{-1}(L_A - qR_B)Y]$$

$$= Tr[X^* m_g(L_A, qR_B)Y] - Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y],$$

$$I_{A,B,q}^{(g,f)}(X) = \Gamma_{A,B,q}^{(g,f)}(X,X),$$

$$\Psi_{A,B,q}^{(g,f)}(X,Y) = Tr[X^* m_g(L_A, qR_B)Y] + Tr[X^* m_{\Delta_g^f}(L_A, qR_B)Y],$$

$$J_{A,B,q}^{(g,f)}(X) = \Psi_{A,B,q}^{(g,f)}(X,X),$$

$$U_{A,B,q}^{(g,f)}(X) = \sqrt{I_{A,B,q}^{(g,f)}(X) J_{A,B,q}^{(g,f)}(X)}.$$

The quantities $I_{A,B,q}^{(g,f)}(X)$ and $\Gamma_{A,B,q}^{(g,f)}(X,Y)$ are said generalized quasi-metric adjusted q skew information and generalized quasi-metric adjusted q correlation measure, respectively.

Theorem 2.1 (Schrodinger type). For $f \in \mathfrak{F}_{op}^r$, it holds

$$I_{A,B,q}^{(g,f)}(X) \cdot I_{A,B,q}^{(g,f)}(Y) \geq |\Gamma_{A,B,q}^{(g,f)}(X,Y)|^2,$$

where $X,Y \in M_n(\mathbb{C})$, $A,B \in M_{n,+}(\mathbb{C})$ and q > 0.

We use only Schwarz inequality to prove Theorem 2.1 similarly to the proof of Theorem 2 in [4]. We note the equation

$$|L_A - qR_B| = \sum_{i=1}^{n}\sum_{j=1}^{n} |\lambda_i - q\mu_j| L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|},$$

where $A = \sum_{i=1}^n \lambda_i |\phi_i\rangle\langle\phi_i|$, $B = \sum_{j=1}^n \mu_j |\psi_j\rangle\langle\psi_j|$ are the spectral decompositions.

Theorem 2.2 (Heisenberg type) For $f \in \mathfrak{F}_{op}^r$, *if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \qquad (2.2)$$

for some $\ell>0$, then it holds

$$U_{A,B,q}^{(g,f)}(X) \cdot U_{A,B,q}^{(g,f)}(Y) \geq k\ell |Tr[X^* |L_A - qR_B| Y]|^2,$$

where $X,Y \in M_n(\mathbb{C})$, $A,B \in M_{n,+}(\mathbb{C})$ and $q>0$. In particular,

$$k\ell(Tr[X^*|L_A - qR_B|X])^2 \qquad (2.3)$$

$$\leq Tr[X^*(m_g(L_A, qR_B) - m_{\Delta_g^f}(L_A, qR_B))X]$$

$$\times Tr[X^*(m_g(L_A, qR_B) + m_{\Delta_g^f}(L_A, qR_B))X],$$

where $X \in M_n(\mathbb{C})$, $A,B \in M_{n,+}(\mathbb{C})$ and $q>0$.

We use refined Schwarz inequality to prove Theorem 2.2 similar to the proof of Theorem 3 in [4].

Trace inequalities

We assume that

$$g(x) = \frac{x+1}{2},\ f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},\ k = \frac{f(0)}{2},\ \ell = 2.$$

Then, since (2.1), and (2.2) are satisfied for g,f,k and ℓ, we have the following trace inequality by putting $X = I$ in (2.3).

$$\alpha(1-\alpha)(Tr[|L_A - qR_B|I])^2 \qquad (3.1)$$

$$\leq \left(\frac{1}{2}Tr[A+qB]\right)^2 - \left(\frac{1}{2}Tr[A^\alpha(qB)^{1-\alpha} + A^{1-\alpha}(qB)^\alpha]\right)^2.$$

This is a generalization of trace inequality given in [8]. And also we give the following new inequality by combining the Chernoff-type inequality with the above theorem.

Theorem 3.1 We have the following:

$$\frac{1}{2}Tr[A+qB-|L_A - qR_B|I] \leq \inf_{0\leq\alpha\leq1} Tr[A^{1-\alpha}(qB)^\alpha]$$

$$\leq Tr[A^{1/2}(qB)^{1/2}] \leq \frac{1}{2}Tr[A^\alpha(qB)^{1-\alpha} + A^{1-\alpha}(qB)^\alpha]$$

$$\leq \sqrt{\left(\frac{1}{2}Tr[A+qB]\right)^2 - \alpha(1-\alpha)(Tr[|L_A - qR_B|I)^2}.$$

We need the following lemma in order to prove Theorem 3.1.

Lemma 3.1 Let $f(s) = Tr[A^{1-s}(qB)^s]$ *for* $A,B \in M_n(\mathbb{C})$, $0 \leq s \leq 1$ and $q > 0$. Then $f(s)$ is convex in s.

Proof of Lemma 3.1.

$f'(s) = Tr[-A^{1-s}\log A(qB)^s + A^{1-s}(qB)^s \log qB]$. And then

$$f''(s) = Tr[A^{1-s}(\log A)^2(qB)^s - A^{1-s}\log A(qB)^s \log qB]$$

$$-Tr[A^{1-s}\log A(qB)^s \log qB - A^{1-s}(qB)^s(\log qB)^2]$$

$$= Tr[A^{1-s}(\log A)^2(qB)^s] - Tr[A^{1-s}\log A \log qB(qB)^s]$$

$$-Tr[\log qB \log A A^{1-s}(qB)^s] + Tr[A^{1-s}(\log qB)^2(qB)^s]$$

$$= Tr[A^{1-s}\log A(\log A - \log qB)(qB)^s]$$

$$-Tr[A^{1-s}(\log A - \log qB)\log qB(qB)^s]$$

$$= Tr[A^{1-s}(\log A - \log qB)(qB)^s \log A]$$

$$-Tr[A^{1-s}(\log A - \log qB)\log qB(qB)^s]$$

$$= Tr[A^{1-s}(\log A - \log qB)(qB)^s(\log A - \log qB)]$$

$$= Tr[A^{(1-s)/2}(\log A - \log qB)(qB)^s(\log A - \log qB)A^{(1-s)/2}] \geq 0.$$

Then $f(s)$ is convex in s.

Proof of Theorem 3.1. The third and fourth inequalities follow from Lemma 3.1 and (3.1), respectively. So we may only prove

$$Tr[A+qB-|L_A - qR_B|I] \leq 2Tr[A^{1-\alpha}(qB)^\alpha] \quad (0 \leq \alpha \leq 1).$$

Let

$$A = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i| = \sum_{i,j}\lambda_i |\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|,$$

$$B = \sum_j \mu_j |\psi_j\rangle\langle\psi_j| = \sum_{i,j}\mu_j |\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|.$$

Then we have

$Tr[A]=\sum_{i,j}\lambda_i|\langle\phi_i|\psi_j\rangle|^2, Tr[B]=\sum_{i,j}\mu_j|\langle\phi_i|\psi_j\rangle|^2.$

And since

$|L_A-qR_B|=\sum_{i,j}|\lambda_i-q\mu_j|L_{|\phi_i\rangle\langle\phi_i|}R_{|\psi_j\rangle\langle\psi_j|},$

we have

$|L_A-qR_B|I=\sum_{i,j}|\lambda_i-q\mu_j||\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|.$

Then we have

$Tr[|L_A-qR_B|I]=\sum_{i,j}|\lambda_i-q\mu_j||\langle\phi_i|\psi_j\rangle|^2.$

Therefore

$Tr[A+qB-|L_A-qR_B|I]=\sum_{i,j}(\lambda_i+q\mu_j-|\lambda_i-q\mu_j|)|\langle\phi_i|\psi_j\rangle|^2.$

On the other hand since

$A^\alpha=\sum_i\lambda_i^\alpha|\phi_i\rangle\langle\phi_i|=\sum_{i,j}\lambda_i^\alpha|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|,$

$B^{1-\alpha}=\sum_j\mu_j^{1-\alpha}|\psi_j\rangle\langle\psi_j|=\sum_{i,j}\mu_j^{1-\alpha}|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|,$

we have

$A^\alpha(qB)^{1-\alpha}=\sum_{i,j}\lambda_i^\alpha(q\mu_j)^{1-\alpha}|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|.$

Then

$Tr[A^\alpha(qB)^{1-\alpha}]=\sum_{i,j}\lambda_i^\alpha(q\mu_j)^{1-\alpha}|\langle\phi_i|\psi_j\rangle|^2.$

Thus

$2Tr[A^\alpha(qB)^{1-\alpha}]-Tr[A+qB-|L_A-qR_B|I]$

$=\sum_{i,j}\{2\lambda_i^\alpha(q\mu_j)^{1-\alpha}-(\lambda_i+q\mu_j-|\lambda_i-q\mu_j|)\}|\langle\phi_i|\psi_j\rangle|^2.$

Since $2x^\alpha(qy)^{1-\alpha}-(x+qy-|x-qy|)\geq 0$ for $x,y>0, 0\leq\alpha\leq 1$ and q>0 in general, we can get the result.

Remark 3.1 There is no relationship between $Tr[|A-qB|]$ and $Tr[|L_A-qR_B|I]$. For example, let

$$A=\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{3}{2}\end{pmatrix},\quad B=\begin{pmatrix}8 & 0\\ 0 & 2\end{pmatrix}$$

and $q=\frac{1}{2}$ Then $Tr[L_A-qR_B|I]=3$ and $Tr[|A-qB|]=\sqrt{10}$.

On the other hand, let

$$A=\begin{pmatrix}\frac{13}{2} & \frac{7}{2}\\ \frac{7}{2} & \frac{13}{2}\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 0 & \frac{5}{2}\end{pmatrix}$$

And q=2. Then $Tr[|L_A-qR_B|I]=8$ and $Tr[|A-qB|]=\sqrt{58}$. Then Theorem 3.1 and trace inequality given by Audenaert et al and Powers-Stφrmer have no relationship.

Conclusion

We gave a non-hermitian q uncertainty relation and apply to the trace inequalities related to fidelity and trace distance for different two generalized states.

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