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Research Article

Adjusted Hardy-Rogers-Type Result Generalization

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Abstract

The adjusted Hardy-Rogers result generalization for the fixed point is demonstrated in this study, validating our results utilizing an application.

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Introduction

The existence and uniqueness of a point $\xi \in X$, such that $T: X \to X$, is a contraction mapping where X is a complete metric space was proved by Banach [1].

$$d(f\xi,f\zeta) \le \alpha d(\xi,\zeta), \tag{1}$$

for all $\xi, \zeta \in X$ and α [0,1). Kannan [2] developed (1) as

$$d(f\xi,f\zeta) \le \alpha[d(f\xi,\xi)+d(f\zeta,\zeta)], \tag{2}$$

for all $\xi,\zeta \in X$ and $\alpha \in (0,\frac{1}{2})$. Reich in [3] generalized (2) as

$$d(f\xi,f\zeta) \le [\eta_1 d(\xi,\zeta)+\eta_2 d(f\xi,\xi)+\eta_3 d(f\zeta,\zeta)], \tag{3}$$

for all $\xi, \zeta \in X$ such that $\eta_1 + \eta_2 + \eta_3 < 1$. Then f has a unique fixed point in X.

In the same direction, Hardy and Rogers in [4] introduced the following

Theorem 1.1

Let (X, d) be a metric space and f a self mapping of X satisfies

$$d(f\xi,f\zeta) \le \eta_1 d(\xi,f\xi)+\eta_2 d(\zeta,f\zeta)+\eta_3 d(\xi,f\zeta)+\eta_4 d(\zeta,f\xi)+\eta_5 d(\xi,\zeta), \tag{4}$$

for $\xi, \zeta \in X$ where $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ are non-negative and we set $\alpha = \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5$. Then,

If X is complete metric space and $\alpha 1$, f has a unique fixed point.

If (4) is adjusted to the condition $\xi \neq \zeta$ implies

$$d(f\xi,f\zeta)\eta_1 d(\xi,f\xi)+\eta_2 d(\zeta,f\zeta)+\eta_3 d(\xi,f\zeta)+\eta_4 d(\zeta,f\xi)+\eta_5 d(\xi,\zeta). \tag{5}$$

Such that X is a compact with continuous mapping and α + *1*, then f has a unique fixed point.

Recently, many of the Hardy-Rogers-type notions have been developed. From these studies, we refer to Rangama [5] established the existence of the Hardy-Rogers-type common fixed point in 2-metric space. With respect to the aiding function, Chifu [6] provided a few fixed point theorems in b-metric space utilizing the Hardy-Rogers type. New Hardy-Rogers-type results have been provided by Patil, et al. [7]. Victoria [8] obtained the P-proximate cyclic contraction in the uniform spaces utilizing the Hardy-Rogers type. Using partially ordered partial metric space, Abbas [9] developed a few fixed point theorems for the Hardy-Rogers type. The common fixed point theorem for T-Hardy-Rogers contraction mapping in a cone metric space was established by Rhymend, et al. in [10]. Saipara generalized some fixed point theorems for Hardy-Rogers-type in metric-like space [11]. Raghavendran, et al. [12] included a recent article relevant to the focused topic.

Main results

We will introduce and prove the adjusting generalization of the Hardy-Rogers type as

Theorem 2.1

Let $\{f_\alpha\}$ be a family continuous self-mappings in a complete metric space X, suppose that

$$d(f_\alpha(\xi), f_\beta(\zeta)) \le \eta_1 d(\xi, f_\alpha(\xi)) + \eta_2 d(\zeta . f_\beta(\zeta)) + \eta_3 d(\xi, f_\beta(\zeta)) + \eta_4 d(\zeta, f_\alpha(\xi))$$

$$+\eta_5 d(\xi,\zeta) \quad (6)$$

for every $\xi, \zeta \in X$, $\xi \neq \zeta$ and $\eta_1,\eta_2,\eta_3,\eta_4,\eta_5 \in \sum_{i=1}^{5}\eta_i 1$. Then $f_\alpha(\xi)$ has a unique fixed point $u_1 \in X$.

Proof. For $\xi_0,\zeta_0 \in X$ take $f_\alpha(\xi_{n-1}) = \xi_n, g_\beta(\zeta_{n-1}) = \zeta_n$,

$$d(x_k, y_k) = d(f_\alpha(\xi_{k-1}, g_\beta(\zeta_{k-1}))$$

$$\le \eta_1 d(\xi_{k-1}, f_\alpha(\xi_{k-1})) + \eta_2 d(\zeta_{k-1}, g(\zeta_{k-1}) + \eta_3 d(\xi_{k-1}, g(\zeta_{k-1})$$

$$+\eta_4 d(\zeta_{k-1}, f_\alpha(\xi_{k-1} + \eta_5 d(\xi_{k-1},\zeta_{k-1})), k \in \mathbb{N}. \quad (7)$$

So,

$$\sum_{k=1}^{n} d(x_k, y_k) = \sum_{k=1}^{n} d(f_{\alpha 1}(\xi_{k-1}, f_{\alpha 2}(\zeta_{k-1}))$$

$$\le \sum_{k=1}^{n} [\eta_1 d(\xi_{k-1},\xi_k) + \eta_2 d(\zeta_{k-1},\zeta_k) + \eta_3 d(\xi_{k-1},\zeta_k) + \eta_4 d(\zeta_{k-1},\xi_k)$$

$$+\eta_5 d(\xi_{k-1},\zeta_{k-1})]$$

$$\le [\eta_1 d(\xi_0,\xi_n) + \eta_2 d(\zeta_0,\zeta_n) + \eta_3 \sum_{k=1}^{n} d(\xi_{k-1},\zeta_k) + \sum_{k=1}^{n} \eta_4 d(\zeta_{k-1},\xi_k)$$

$$+\sum_{k=1}^{n} \eta_5 d(\xi_{k-1},\zeta_{k-1})].$$

Also,

$$\sum_{k=1}^{n} d(\xi_{k+1}, y_k) \le [\eta_1 d(\xi_1,\xi_n) + \eta_2 d(\zeta_0,\zeta_n) + \eta_3 \sum_{k=1}^{n} d(\xi_k,\zeta_k) + \sum_{k=1}^{n} \eta_4 d(\zeta_{k-1},\xi_k)$$

$$+\sum_{k=1}^{n} \eta_5 d(\xi_{k-1},\zeta_{k-1})],$$

and,

$$\sum_{k=1}^{n} d(\xi_k,\xi_{k+1}) \le (\eta_1 + \eta_5) d(\xi_0,\xi_n) + (\eta_2 + \eta_3) d(\xi_1,\xi_{n+1}).$$

Then,

$$\sum_{k=1}^{n} d(\xi_k,\xi_{k+1}) \le \sum_{k=1}^{n} d(\xi_k, y_k) \le \sum_{k=1}^{n} d(\xi_{k+1}, y_k) \infty. \quad (8)$$

Therefore, $\sum_{k=1}^{n} d(\xi_k,\xi_{k+1}) \to 0$ *as* $k \to \infty$, hence $\{X_K\}$ is a Cauchy sequence. Also, $\{Y_K\}$ is a Cauchy sequence in X, and since X is a complete metric space, there exists a common fixed point in X.

Suppose that,

$$u_1 = \lim_{n\to\infty} \xi_n, \quad u_2 = \lim_{n\to\infty} \zeta_n, \quad \forall u_1, u_2 \in X,$$

we get,

$$d(\xi_n, u_1) \to 0, \quad n \to \infty,$$

$$d(\zeta_n, u_1) \to 0, \quad n \to \infty.$$

since, f_α, g_β are continuous mappings we obtained,

$$d(f_\alpha(\xi_n), f_\alpha(u_1)) \to 0, \quad n \to \infty,$$

$$d(g_\beta(\zeta_n), g_\beta(u_2)) \to 0, \quad n \to \infty.$$

We have

$$d(u_1, f_\alpha(u_1)) = d(f_\alpha^{-1}(f_\alpha(u_1)), f_\alpha(u_1))$$

$$\le \eta_1 d(f_\alpha^{-1}(f_\alpha(u_1)), f_\alpha(u_1)) + \eta_2 d(u_1, f_\alpha(u_1)) + \eta_3 d(f_\alpha(u_1), f_\alpha(u_1))$$

$$+\eta_4 d(u_1, f_\alpha^{-1}(f_\alpha(u_1))) + \eta_5 d(f_\alpha(u_1), u_1)$$

$$= (\eta_1 + \eta_2 + \eta_5) d(u_1, f_\alpha(u_1)).$$

Hence, $f_\alpha(u_1) = u_1$.

likewise, we can prove that $g_\beta(u_2) = u_2$. Now, we will prove that u_1 is a common fixed point of f_α and g_β, as

$$d(u_1,u_2) \le \eta_1 d(u_1, f_{\alpha 1}(u_1)) + \eta_2 d(u_2, f_{\alpha 2}(u_2)) + \eta_3 d(u_1, f_{\alpha 2}(u_2)) + \eta_4 d(u_2, f_{\alpha 1}(u_1))$$

$$+\eta_5 d(u_1,u_2)$$

$$= (\eta_3 + \eta_4 + \eta_5) d(u_1,u_2).$$

Consider $u_{3 \in X}$ such that it can be used to demonstrate the uniqueness of u_1.

$$f_{\alpha(}u_3) = u_3, \quad and \quad g_\beta(u_3) = u_3.$$

Therefore

$$d(u_1,u_3) = (f_{\alpha 1}(u_1), f_{\alpha 3}(u_3))$$

$$\le \eta_1 d(u_1, f_{\alpha 1}(u_1)) + \eta_2 d(u_3, f_{\alpha 2}(u_3)) + \eta_3 d(u_1, f_{\alpha 2}(u_3))$$

$$+\eta_4 d(u_3, f_{\alpha 1}(u_1)) + \eta_5 d(u_1,u_3)$$

$$= (\eta_3 + \eta_4 + \eta_5) d(u_1,u_3).$$

Hence,

$$u_1 = u_2 = u_3.$$

Thus, u_1 is the unique fixed point of f_α and g_β.

Theorem 2.1 can be stated as follows:

Theorem 2.2

Let f_k be a self-mappings on X, such that $f_k(z_k) = z_k, \forall \xi \in X$ and $z_k \in X$ $\forall k$ respectively, such that

$$d(f_k(\xi), f_k(\zeta)) \le \eta_1 d(\xi, f_k(\xi)) + \eta_2 d(\zeta, f_k(\zeta)) + \eta_3 d(\xi, f_k(\zeta)) + \eta_4 d(\zeta, f_k(\xi))$$

$$+\eta_5 d(\xi,\zeta). \quad (9)$$

For all $\xi, \zeta \in X$, $\xi \neq \zeta$ and $\sum_{i=1}^{5}\eta_i 1$.

Proof. Theorem 2.1 may be proven using the same way used to prove Theorem 2.2.

Our main result has corollaries, we leave their proof for the reader.

Corollary 2.3

Let X be a complete metric space and let $f: X \to R$ a continuous *self-mapping on X, let f satisfying (4) for all ξ, ζ ∈ X, ξ ≠ ζ and for some* $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5 \in [0,1)$ *such that* $\sum_{i=1}^{5}\eta_i$. *Then f has a unique fixed point.*

Corollary 2.4

Let X be a complete metric space and let f, g are two continuous self-mappings on X satisfying

$$d(f(\xi),g(\zeta)) \le \eta_1 d(\xi,f(\xi)) + \eta_2 d(\zeta,g(\zeta)) + \eta_3 d(\xi,g(\zeta))$$

$$+\eta_4 d(\zeta,f(\xi)) + \eta_5 d(\xi,\zeta) \quad (10)$$

for all ξ, ζ ∈ *X*, ξ ≠ ζ and for some $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5 \in [0,1)$ such that $\sum_{i=1}^{5}\eta_i 1$. Then *f* and *g* have a unique fixed point.

The existence and uniqueness of a common fixed point of two mappings that are not necessarily continuous can be investigated using our findings by introducing the next theorem [13-15].

Theorem 2.5

Let $f\alpha_1, f\alpha_2$ *be two self-mappings on a complete metric space X, satisfies*

$$d(f\alpha_1(\xi), f\alpha_2(\zeta)) \le \eta_1(\xi, f\alpha_1(\xi)) + \eta_2(\zeta, f\alpha_2(\zeta)) + \eta_3(\xi, f\alpha_2(\zeta)) + \eta_4(\zeta, f\alpha_1(\xi))$$

$$+\eta_5(\xi,\zeta),$$

for all ξ, ζ ∈ *X*, ξ ≠ ζ and $\sum_{i=1}^{5}\eta_i 1$. Suppose that $f\alpha_1 f\alpha_2 = f\alpha_2 f\alpha_1$ is continuous then $f\alpha_1$ and $f\alpha_2$ having a unique common fixed point in ξ.

Proof. Take

$$\xi_n = f\alpha_1(\xi_{n-1}), \xi_n = f\alpha_2(\xi_{n-1}) \quad and \quad f\alpha_1(\xi_{n-1}) \ne f\alpha_2(\xi_{n-1}), \xi_n \ne \xi_{n-1}, \forall n \in \mathbb{N}.$$

Therefore,

$$d(\xi_{2n+1}, \xi_{2n}) = d(f\alpha_1(\xi_{2n}), f\alpha_2(\xi_{2n-1}))$$

$$\le \eta_1(\xi_{2n}, f\alpha_1(\xi_{2n})) + \eta_2(\xi_{2n-1}, f\alpha_2(\xi_{2n-1})) + \eta_3(\xi_{2n}, f\alpha_2((\xi_{2n-1}))$$

$$+\eta_4((\xi_{2n-1}, f\alpha_1(\xi_{2n})) + \eta_5(\xi_{2n}, \xi_{2n-1})$$

$$= \eta_1(\xi_{2n}, \xi_{2n+1}) + \eta_2(\xi_{2n-1}, \xi_{2n}) + \eta_3(\xi_{2n}, \xi_{2n}) + \eta_4(\xi_{2n-1}, \xi_{2n+1})$$

$$+\eta_5(\xi_{2n}, \xi_{2n-1}).$$

So, we have

$$d(\xi_{2n+1}, \xi_{2n}) \le \left(\frac{\eta_2 + \eta_4 + \eta_5}{1 - \eta_2 - \eta_4}\right) d(\xi_{2n}, \xi_{2n-1}). \quad (11)$$

From (11) we obtain

$$d(\xi_{2n+1}, \xi_{2n}) \le \left(\frac{\eta_2 + \eta_4 + \eta_5}{1 - \eta_2 - \eta_4}\right)^{2n} d(\xi_1, \xi_0). \quad (12)$$

We get

$$f\alpha_1 f\alpha_2(u_1) = f\alpha_2 f\alpha_1(u_1) = f\alpha_1 f\alpha_2(\lim_{k\to\infty}\xi_{n_k}) = \lim_{k\to\infty}\xi_{n_{k+1}} = u_1.$$

Let u_1 is a fixed point of $f_1 f\alpha_2$ such that $f\alpha_1 f\alpha_2(u_1) = u_1$. Now, we must show that $f\alpha_1(u_1) = u_1$ and $f\alpha_2(u_1) = u_1$. For that we let $f\alpha_1(u_1) \ne u_1$ and $f\alpha_2(u_1) \ne u_1$. Then,

$$d(u_1, f\alpha_1(u_1)) = d(f\alpha_2 f\alpha_1(u_1), f\alpha_1(u_1))$$

$$\le \eta_1 d(f\alpha_1(u_1), f\alpha_2 f\alpha_1(u_1) + \eta_2 d(u_1, f\alpha_1(u_1)) + \eta_3 d(f\alpha_1(u_1, f_1(u_1))$$

$$+\eta_4 d(u_1, f\alpha_1(f\alpha_1(u_1))) + \eta_5 d(f\alpha_1(u_1), u_1) \;\; = 0.$$

Hence,

u_1 is a fixed point of $f\alpha_1$. Similarly we can get $f\alpha_2(u_1) = u_1$. This indicates that $f\alpha_1$ and $f\alpha_2$ have a common fixed point in *X*. That was proof of existence.

As for proving uniqueness, let's suppose $u_2 \in X$, $u_2 \ne u_1$ be another fixed point of f_1 and $f\alpha_2$. Then

$$d(u_1, u_2) = d(f\alpha_1(u_1), f\alpha_2(u_2))$$

$$\le \eta_1 d(u_1, f\alpha_1(u_1)) + \eta_2 d(u_2, f\alpha_2(u_2)) + \eta_3 d(u_1, f\alpha_2(u_2) +$$

$$\eta_4 d(u_2, f\alpha_1(u_1) + \eta_5 d(u_1, u_2)$$

$$= (\eta_3 + \eta_4 + \eta_5) d(u_1, u_2)$$

$$= 0.$$

We have demonstrated a uniqueness and completed proof of the theorem.

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