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Mini Review

Approximation of Kantorovich-type Generalization of (*p*,*q*) - Bernstein type Rational Functions Via Statistical Convergence

Hayatem Hamal*

Department of Mathematics, Tripoli University, Tripoli 22131, Libya

Abstract

In this paper, we use the modulus of continuity to study the rate of A-statistical convergence of the Kantorovich-type (p,q) - analogue of the Balázs–Szabados operators by using the statistical notion of convergence.

Mathematics subject classification: Primary 4H6D1; Secondary 4H6R1; 4H6R5

More Information

***Address for correspondence:** Hayatem Hamal, Department of Mathematics, Tripoli University, Tripoli 22131, Libya, Email: hafraj@yahoo.com

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Introduction

Bernstein type rational functions, $R_n(f;x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_nx)^k \quad (n=1,2,\ldots)$ Balázs defined and investigated them in 1975, (see [1]). In this definition, *f* is a real and single-valued function defined on the interval [0,∞), a_n and b_n are real numbers that have been appropriately chosen and are independent of *x*. Seven years later, in 1982, Balázs and Szabados cooperated to improve the estimate in [1] by selecting appropriate parameters a_n and b_n under some restrictions for *f*(*x*), (see[2]).

Recently, different *q* - generalizations of Balázs-Szabados operators have been studied by several researchers, see [3-7]. In [8], the Kantorovich-type *q* - analogue of the Balázs-Szabados operators is defined by Hamal and Sabancigil as follows:

$$R^*_{n,q}(f,x)=\sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 f\left(\frac{[k]_q+q^k t}{b_n}\right)d_qt, \quad (1)$$

where

$$f:[0,\infty)\to\mathbb{R},\ q\in(0,1),\ a_n=[n]_q^{\beta-1},$$

$$b_n=[n]_q^{\beta},\ 0<\beta\le\frac{2}{3},\ n\in\mathbb{N},\ x\ge 0,$$

and $r_{n,k}(q,x)=\frac{1}{(1+a_nx)^n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a_nx)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right).$

Additionally, the fast rise of (*p*,*q*) - calculus has encouraged many mathematicians in this subject to discover different generalizations. In the last decade, Mursaleen et al. defined and studied the analogue of many operators (see [9-15]). The (*p*,*q*) - generalization of Szász–Mirakjan operators was studied by Acar (see [16]), (*p*,*q*) - Kantorovich modification of Bernstein operators was studied by Acar and Aral (see [17]).

In [18-20], recently, Hamal and Sabancigil introduced a new Kantorovich-type (p,q) - analogue of the Balázs–Szabados operators by generalizing the new Kantorovich-type q - analogue of Balázs–Szabados operators, given by (1), as follows:

$$R^{*}_{n,p,q}\left(f,x\right)=\sum_{k=0}^{n} r^{*}_{n,k}\left(p,q,x\right)\int_{0}^{1} f\left(\frac{p^{n-k}\left(\left[k\right]_{p,q}+q^{k}t\right)}{b_n}\right)d_{p,q}t\,, \quad (2)$$

where $r^{*}_{n,k}\left(p,q,x\right)=\frac{1}{p^{n(n-1)/2}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2}\left(\frac{a_n x}{1+a_n x}\right)^{k}\prod_{j=0}^{n-k-1}\left(p^{j}-q^{j}\frac{a_n x}{1+a_n x}\right)$

and $0<q<p\le 1,\ a_n=\left[n\right]_{p,q}^{\beta-1},\ b_n=\left[n\right]_{p,q}^{\beta},\ 0<\beta\le\frac{2}{3},\ n\in\mathbb{N},\ x\ge 0,\ f:\left[0,\infty\right)\to\mathbb{R}.$

These newly defined operators have some advantages when they are compared with the other (p,q) - analogues given in the other studies. The first advantage is that they are positive for all continuous and real-valued functions on the half-open interval $[0,\infty)$. The second advantage is that they can be used to approximate also the integrable functions. If $p = 1$, these polynomials reduce to the new Kantorovich-type analogue of the Balázs–Szabados operators, which are defined by Hamal and Sabancigil in [8]. Moreover, we considered the following two special casea:

- If $0<p<q\le 1$ or $1\le p<q<\infty$ or when the positivity property of the operators fails.
- If $1\le q<p<\infty$ then approximation by the new operators $R^{*}_{n,p,q}\left(f,x\right)$ becomes difficult because if p is large enough then the sequence $\left\{R^{*}_{n,p,q}\right\}_{n\in\mathbb{N}}$ may diverge.

Before stating the main result for these operators, we give some notations and definitions of (p,q) - calculus. For any $p>0$, $q>0$ non-negative integer n, the (p,q) - integer of the number n is defined as follows:

$$\left[n\right]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^{2}+\ldots+pq^{n-2}+q^{n-1}=\begin{cases}\dfrac{p^{n}-q^{n}}{p-q} & if\ p\ne q\ne 1\\ np^{n-1} & if\ p=q\ne 1\\ \left[n\right]_{q} & if\ p=1\\ n & if\ p=q=1\end{cases},$$

$$\left[n\right]_{p,q}!=\prod_{k=1}^{n}\left[k\right]_{p,q},\ n\ge 1 \ \text{ and }\ \left[0\right]_{p,q}!=1,$$

and (p,q) - binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{\left[n\right]_{p,q}!}{\left[k\right]_{p,q}!\left[n-k\right]_{p,q}!},\ 0\le k\le n.$$

The formula of (p,q) - binomial expansion is defined by

$$\left(ax+by\right)^{n}_{p,q}=\sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k}b^{k}x^{n-k}y^{k}=\left(ax+by\right)\left(pax+qby\right)\left(p^{2}ax+q^{2}by\right)\ldots\left(p^{n-1}ax+q^{n-1}by\right).$$

Let $f:C[0,a]\to\mathbb{R}$, the (p,q) - integral of is defined by:

$$\int_{0}^{a} f(t)d_{p,q}t=(p-q)a\sum_{k=0}^{\infty} f\left(\frac{q^{k}}{p^{k+1}}a\right)\frac{q^{k}}{p^{k+1}} \text{ if } \left|\frac{p}{q}\right|>1\,.$$

Fast [21] and Fridy [22] provided the following notions.

Suppose that $E\subseteq\mathbb{N}=\left\{1,\ 2,\ldots\right\}$ and $E_n=\left\{k\le n:k\in E\right\}$. Then $\delta\left(E\right)=\lim_{n\to\infty}\frac{1}{n}\left|E_n\right|$ is called the natural density of E provided that the limit exists.

Definition 1: A sequence $x = (x_n)$ is statistically convergent to the number L if for every $\varepsilon>0$, we have $\delta\left\{k\in\mathbb{N}:\left|x_k-L\right|\ge\varepsilon\right\}=0$ is denoted by $st_A-\lim_{n\to\infty} x_n=L$.

Because all finite subsets of the natural numbers have density zero, any convergent sequence is statistically convergent, but not contrariwise.

For example, consider the sequence $A = \{a_{n,} n = 1,2,3\ldots\}$ whose terms are

$$a_n = \begin{cases} \sqrt{n} & \text{when } n = m^2, \ \forall m = 1, 2, 3, ... \\ 1 & \text{otherwise} \end{cases}$$

We can see that the sequence is divergent in the ordinary sense, but it is statistically convergent to 1.

Let $C_B[a,b]$ denote the space of all functions *f* which are continuous in every point of the interval [*a*,*b*] and bounded on the entire positive real line, $|f(x)| \le M_f, \forall x \in (0,\infty)$.

Lemma 1 ([10]): For all Let $n \in \mathbb{N},\ x \in [0,\infty)$ and 0<*q*<*p*≤1, we have the following equalities:

$$R^*_{n,p,q}(1,x) = 1.$$

$$R^*_{n,p,q}(t,x) = \frac{p^n}{[2]_{p,q} b_n} + \frac{2q}{[2]_{p,q}}\left(\frac{x}{1+a_n x}\right).$$

$$R^*_{n,p,q}(t^2,x) = \frac{p^{2n}}{[3]_{p,q} b_n^2} + \frac{\left(4q^3 + 5q^2 p + 3qp^2\right)p^{n-1}}{[2]_{p,q}[3]_{p,q} b_n}\left(\frac{x}{1+a_n x}\right)$$
$$+ \frac{q[n-1]_{p,q}}{[n]_{p,q}} \frac{4q^3 + q^2 p + qp^2}{[2]_{p,q}[3]_{p,q}}\left(\frac{x}{1+a_n x}\right)^2.$$

Lemma 2 ([10]): For all $n \in \mathbb{N},\ x \in [0,\infty)$ <*q*<*p*≤1, we have the following estimations:

$$\left(R^*_{n,p,q}((t-x),x)\right)^2 \le \frac{1}{b_n}\left\{\frac{1}{b_n} + \frac{\left(p^n - q^n\right)^2}{b_n}\left(\frac{1}{p+q} + \frac{1}{p-q}(a_n x)\right)^2\right\},\ x \in [0,\infty),\ (3)$$

$$R^*_{n,p,q}\left((t-x)^2,x\right) \le \frac{A_1}{b_n}\phi_n(p,q)(1+x)^2,\quad x \in [0,\infty),\ (4)$$

$$R^*_{n,p,q}\left((t-x)^4,x\right) \le \frac{A_2}{b_n^2}(1+x)^2,\ x \in [0,\infty),\ (5)$$

Where

$A_1 > 0, A_2 > 0$ and $\phi_n(p,q) = \max\left\{p^{n-1},\ b_n - a_n p^{n-1},\ \frac{1}{[3]_{p,q} b_n}\right\}.$

In the following theorem, the Bohman –Korovkin type statistical approximation theorem was proved by Gadjiev and Orhan [23].

Theorem 1 ([13]): Let $(\ell_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting from $C_B[a,b]$ to $B[a,b]$ that is, $\ell_n : C_B[a,b] \to B[a,b]$ satisfies the conditions that

$$st_A - \lim \left\|\ell_n(e_i) - e_i\right\| = 0 \text{ with } e_i(t) = t^i \text{ and } \forall\ i = 0,\ 1, 2.\ (6)$$

Then, we have

$$st_A - \lim_n \left\|\ell_n f - f\right\| = 0\ ,\forall\ f \in C_B([a,b]).$$

Now, we give the main result of this research is to use the modulus of continuity to study the rate of A-statistical convergence of Kantorovich-type (*p*,*q*) - analogue of the Balázs–Szabados operators $R^*_{n,p,q}(f,x)$.

Theorem 2: Let $q = (q_n), p = (p_n),\ 0 < q_n < p_n \le 1$ such that $st_A - \lim_n q_n = 1, st_A - \lim_n p_n = 1$ and $st_A - \lim_n p_n^n = 1$. Then for each compact interval $[0,b] \subset [0,\infty)$, we have $st_A - \lim_n \left\|R^*_{n,p,q}(f,x) - f(x)\right\| = 0,\ \forall\ f \in C([0,b])$.

Proof: According to Theorem 1, it is sufficient to show that it satisfies (6). By using Lemma 1, it is clear that

$$st_A - \lim_n \left\| R^*_{n,p_n,q_n}(e_0;x) - e_0 \right\| = 0 \text{, since } R^*_{n,p_n,q_n}(e_0;x) = 1. \ (7)$$

Again by Lemma 1, we have

$$\left| R^*_{n,p_n,q_n}(e_1;x) - e_1 \right| = \left| \frac{p_n^n}{[2]_{p,q} b_n} + \frac{2q_n}{[2]_{p_n,q_n}} \left(\frac{x}{1+a_{n,p_n,q_n}x} \right) - x \right|$$
$$= \frac{p_n^n}{[2]_{p_n,q_n} b_n} + \frac{(p_n - q_n)}{[2]_{p_n,q_n}} \frac{x}{1+a_{n,p_n,q_n}x} + \frac{a_{n,p_n,q_n}x^2}{1+a_{n,p_n,q_n}x}.$$

By taking the maximum of both sides of the last equality on [0,*b*] with $0 < b < \frac{1}{a_{n,p_n,q_n}}$, we obtain

$$\left\| R^*_{n,p_n,q_n}(e_1;x) - e_1 \right\| \le \frac{p_n^n}{[2]_{p_n,q_n} b_n} + \frac{(p_n - q_n)}{[2]_{p_n,q_n}} \frac{b}{1+a_{n,p_n,q_n}b} + \frac{a_{n,p_n,q_n}b^2}{1+a_{n,p_n,q_n}b}.$$

By using the limits $st_A - \lim_n q_n = 1, st_A - \lim_n p_n = 1$, we have

$$st_A - \lim_n \frac{p_n^n}{[2]_{p_n,q_n} b_{n,p_n,q_n}} = 0, st_A - \lim_n \frac{(p_n - q_n)}{[2]_{p_n,q_n}} = st_A - \lim_n a_{n,p_n,q_n} = 0,$$

herefore,

$$\left\| R^*_{n,p_n,q_n}(e_1;x) - e_1 \right\| < \varepsilon.$$

For ε>0, we define the sets

$$A := \left\{ n \in N : \left\| R^*_{n,p_n,q_n}(e_1;.) - e_1 \right\| \ge \varepsilon \right\}, \ (8)$$

$$A_1 = \left\{ n \in \mathbb{N} : \frac{p_n^n}{[2]_{p_n,q_n} b_n} \ge \varepsilon \right\}, A_2 = \left\{ n \in \mathbb{N} : \frac{(p_n - q_n)}{[2]_{p_n,q_n}} \frac{b}{1+a_{n,p_n,q_n}b} \ge \varepsilon \right\}, \text{ and}$$

$$A_3 = \left\{ n \in \mathbb{N} : \frac{a_{n,p_n,q_n}b^2}{1+a_{n,p_n,q_n}b} \ge \varepsilon \right\}, \text{ thus from (8), we can see that } A \subseteq A_1 \cup A_2 \cup A_3,$$

$$\delta\left\{ n \in N : \left\| R^*_{n,p_n,q_n}(e_1;.) - e_1 \right\| \ge \varepsilon \right\} \le \delta\left\{ n \in \mathbb{N} : \frac{p_n^{n-1}}{b_{n,p_n,q_n}} \frac{b}{1+a_{n,p_n,q_n}b} \ge \frac{\varepsilon}{3} \right\}$$
$$+ \delta\left\{ n \in \mathbb{N} : \left(1 - \frac{1}{\left(1+a_{n,p_n,q_n}b\right)^2} \right) b^2 \ge \frac{\varepsilon}{3} \right\}$$

$$+ \delta\left\{ n \in \mathbb{N} : \frac{p_n^{n-1`}}{[n]_{p_n,q_n}} \frac{b^2}{\left(1+a_{n,p_n,q_n}b\right)^2} \ge \frac{\varepsilon}{3} \right\}. \ (9)$$

By taking the limit of both sides of the above inequality (9), It is obvious that

$$st_A - \lim_n \frac{p_n^{n-1}}{b_{n,p_n,q_n}} \frac{b}{1+a_{n,p_n,q_n}b} = 0, \ st_A - \lim_n \frac{1}{\left(1+a_{n,p_n,q_n}b\right)^2} = 1, \ st_A - \lim_n \frac{p_n^{n-1`}}{[n]_{p_n,q_n}} \frac{b^2}{\left(1+a_{n,p_n,q_n}b\right)^2} = 0.$$

Which implies

$$st_A - \lim_n \left\| R^*_{n,p_n,q_n}\left(e_1;x\right) - e_1 \right\| = 0\,.\ (10)$$

Also, by using Lemma 1, we may write

$$\left| R^*_{n,p_n,q_n}\left(e_2;x\right) - e_2 \right| \le \left| \frac{p_n^{2n}}{[3]_{p_n,q_n} b_n^2} + \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n} b_n} \left(\frac{x}{1+a_{n,p_n,q_n}x} \right) \right.$$
$$\left. + \frac{q_n [n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} \left(\frac{x}{1+a_{n,p_n,q_n}x} \right)^2 - x^2 \right|$$

$$\le \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} + \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{x}{1+a_n x} \right)$$

$$+ \left\{ 1 - \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} \frac{1}{\left(1+a_{n,p_n,q_n}\right)^2} \right\} x^2 + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} \left(\frac{x}{1+a_n x} \right)^2$$

By taking the maximum of both sides of the last equality on [0,b] with $0 < b < \frac{1}{a_{n,p_n,q_n}}$, we get

$$\left\| R^*_{n,p_n,q_n}\left(e_2;x\right) - e_2 \right\| \le \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} + \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{b}{1+a_{n,p_n,q_n}b} \right)$$

$$+ \left\{ 1 - \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} \frac{1}{\left(1+a_{n,p_n,q_n}b\right)^2} \right\} b^2 + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} \left(\frac{b}{1+a_{n,p_n,q_n}b} \right)^2$$

By using the limits $st_A - \lim_n q_n = 1$, $st_A - \lim_n p_n = 1$, we have

$$st_A - \lim_n \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} = 1,\ st_A - \lim_n \frac{p_n^{n-1}}{[n]_{p_n,q_n}} = 0,\ st_A - \lim_n \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} = 0.$$

Therefore,

$$\left\| R^*_{n,p_n,q_n}\left(e_2;x\right) - e_2 \right\| < \varepsilon.$$

Now, for given ε > 0, we introduce the following sets;

$$D := \left\{ n \in N : \left\| R^*_{n,p_n,q_n}\left(e_2;.\right) - e_2 \right\| \ge \varepsilon \right\},$$

$$D_1 = \left\{ n \in \mathbb{N} : \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} \ge \frac{\varepsilon}{4} \right\},$$

$$D_2 = \left\{ n \in \mathbb{N} : \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{b}{1+a_{n,p_n,q_n}b} \right) \ge \frac{\varepsilon}{4} \right\},$$

Then, from (11) we may write $D \subseteq D_1 \cup D_2 \cup D_3 \cup D_4$,

$$\delta\left\{n \in N : \left\| R^*_{n,p_n,q_n}\left(e_2;.\right) - e_2 \right\| \geq \varepsilon\right\} \leq \delta\left\{n \in \mathbb{N} : \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} \geq \frac{\varepsilon}{4}\right\}$$

$$+\delta\left\{n \in \mathbb{N} : \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n}[3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b}\right) \geq \frac{\varepsilon}{4}\right\}$$

$$+\delta\left\{n \in \mathbb{N} : \left\{1 - \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} \frac{1\,(11)}{\left(1 + a_{n,p_n,q_n} b\right)^2}\right\} b^2 \geq \frac{\varepsilon}{4}\right\}$$

$$+\delta\left\{n \in \mathbb{N} : \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n}[3]_{p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b}\right)^2 \geq \frac{\varepsilon}{4}\right\},$$

by taking the limit of both sides of the above inequality, It is obvious that

$\delta(D) \leq \delta(D_1) + \delta(D_2) + \delta(D_3) + \delta(D_4) = 0$, which implies

$st_A - \lim_n \left\| R^*_{n,p_n,q_n}\left(e_2;x\right) - e_2 \right\| = 0.$ As a result, Equation (6) is proven, yielding the desired result.

Conclusion

In this paper, by using the notion of (p,q) - calculus and statistical convergence, we give the main result of this research to use the modulus of continuity to study the rate of A-statistical convergence of Kantorovich type (p,q) - analogue of the Balázs–Szabados operators.ases:

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